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Lattice scale-free networks with weighted linking

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Recently, models of the scale-free (SF) networks on lattices were investigated, which consider the influence of the embedded space on the networks. Since a lot of real networks exist on the 2D global surface, it is helpful to discuss these models. In this paper, based on the lattice SF networks model, a linking weight is added, thus an additional parameter which can control the clustering coefficient is introduced. Depending on the linking weight, the properties of the model change smoothly from the lattice SF model to the SF random graphs.

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The complex networks provide an effective description for many real systems in nature and society [1–3]. In the last decade, many aspects and objects of complex networks have been studied [4–8] (for more comprehensive knowledge, see reviews on the subject [9]). It has been revealed that many of the real networks such as Internet, co-author networks, metabolic networks [1] etc., hold a scale-free (SF) degree distribution, $P(k)\sim k^{-\lambda}$. Many SF network models have been proposed and deeply studied, including the generalized random graphs [4], the growing Barabasi-Albert (BA) model [5] and vast of its variations (see Ref. [9], and references there), and the lattice embedded models of SF networks [10,11]. Since human's actions always occur in the global surface, considering the lattice models becomes helpful. Recently, Rozenfeld, Cohen, ben-Avraham, and Havlin considered that the spatial distance can affect the connection between the nodes, and proposed a lattice embedded scale-free (LESF) network model [10]. Rozenfeld *et al.* give a rectangle 2D lattice of the size $L\times L$, with the periodic boundary conditions. For each site, they assign a random degree k taken from the scale-free distribution, $P(k)\sim k^{-\lambda}$, $m<k<K$. Then select a site at random and connect it to its closest neighbors until its previously assigned degree k is realized, or up to a cutoff distance $A_c\sqrt{k}$ (note that linking to its neighboring sites is not always possible, in case that the degree quota of the target site is already filled). This process is repeated for all sites of the lattice. Here, the clustering coefficients and average network distances of LESF model are shown by simulation in Fig. 1, cutoff is set to be $A_c=7$. Figure 1(a): the clustering coefficient vs network size. For large networks, they tend to reach a constant value, and for small λ, these networks have large clustering coefficients. Figure 1(b): the average distance vs network size. In most cases the dependence of average distance on network size is of logarithm form, while in certain cases (for large λ) the dependence has a form of $\sim\sqrt{N}$.

In this paper, a weighted linking function is added to LESF model (WLESF). The model is generated as follows: (1) a lattice with periodic boundary conditions of size $L\times L$ is assumed, upon which the network will be embedded; (2) for each node an integer k is assigned as the largest degree it could have, keeping that the distribution of k is a power law function: $P(k)\sim k^{-\lambda}$, $m<k<K$; (3) a node is randomly selected (say, i, with degree k_i) from the lattice, and according to a Gaussian weight function

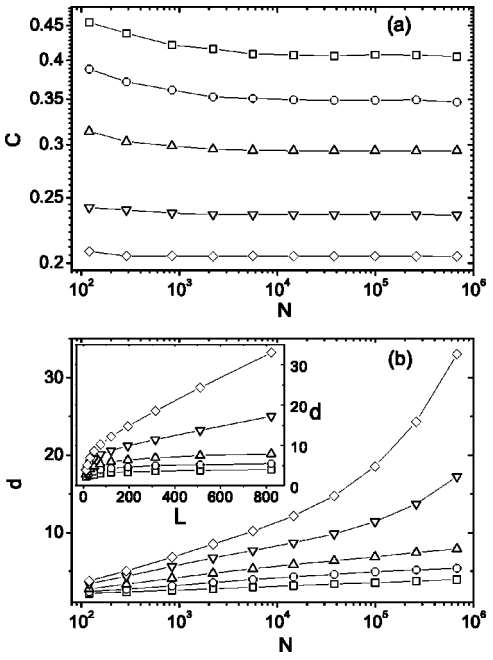

FIG. 1. Properties of LESF networks: (a) clustering coefficients of networks with different power law exponents (λ=2.2 for squares, 2.5 for circles, 3.0 for up triangles, 4.0 for down triangles, and 5.0 for diamonds) and different sizes, as the x axis shows; (b) average distances, the same symbols as those in (a) represent the same values of λ. Inset of (b): average distance d vs lattice side length L.

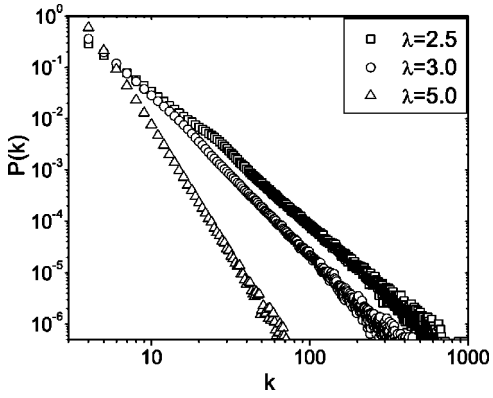

FIG. 2. Degree distribution of the WLESF model with network size $N=10^5$, and $A=1$, the most restricted situation in the simulation, and of different values of λ.

$$f_i(r) = De^{-(r/R_i)^2}, \tag{1}$$

it selects other nodes (say, j) and establishes a connection between them if j's degree quota is not filled yet and there exists no previous connection between i and j, until its degree quota k_i is filled or until it has tried $1/f_i(3R_i)$ times [12], the latter case could be caused by saturation, that is, almost all its spatial neighbors have already fulfilled their degree quotas; (4) the process is repeated throughout all the nodes on the lattice. The normalization constant D, defined by $\int_1^\infty dr 2\pi r f_i(r)=1$, is $(\pi A^2 k_i)^{-1}e^{1/A^2 k_i}$, and $R_i=A\times\sqrt{k_i}$, serves as the characteristic radius of the region that node i can almost freely connect. The cutoff parameter A influences the tightness of local clusters and therefore the topological properties of the networks. The Gaussian form of the weight function is chosen because it is ubiquitous in many natural processes and scientific fields such as physics, biology, statistics, etc. Moreover, it puts natural cutoffs on the model, not the $3R_i$ cutoff, but a more constrained one, as will be discussed later (Fig. 8).

Using computer simulation, we discuss the properties of the WLESF model, such as the average network distance $d=2/N(N-1)\Sigma_{k,l}d_{k,l}$, where $d_{k,l}$ is the network distance between node k and l, say, the number of edges of the shortest path between them; the clustering coefficient $C=(1/N)\Sigma_i(2E_i/k_i(k_i-1))$, where k_i is the edge degree of node i, and E_i is the number of edges between its neighbors; the degree distribution, etc. As shown in Fig. 2, the power law degree distributions of WLESF model are preserved, since all the values of λ are greater than 2 [10].

The direct chemical shell [10] structure is shown in Fig. 3, with different gray levels depicting different shells consisting of nodes with the same network distance from a given node, which is assumed to be the central node in each graph. Along with WLESF, the LESF model and the scale-free random model (SFR) are also presented. Here, λ is fixed to 3.0 for all the three models, and A is varied as 1,2,3, and 5 in WLESF. As A goes larger, the shell boundary blurs and finally disappears as that of random graphs. The shell graph of WLESF shows an obvious transition from the LESF model to the SFR model.

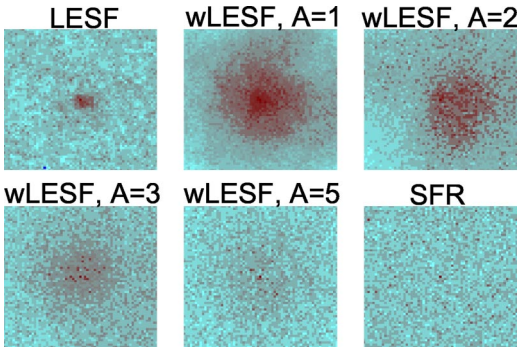

FIG. 3. (Color online) Chemical shells of WLESF model, together with those of LESF model and SFR model. Each shell graph has a size of 195×195.

In Fig. 4, the clustering coefficients of the three models are shown for $\lambda=2.5,3.0,5.0$ in (a), (b), and (c), respectively. For both the LESF model and the WLESF model, the clustering coefficients tend to reach a constant value, which varies with different parameter values of λ and A. For each given degree exponent λ, the larger the A is, the looser the local cluster is, as a result, the smaller the clustering coefficient. For each fixed cutoff parameter A, as λ goes smaller, there are more nodes with large degrees on average, and since any two nodes which have common network neighbors must be spatial neighbors themselves, so the probability that the two nodes connect each other is larger, hence the value of clustering coefficients goes up. As in WLESF, the parameter A can vary continuously, so we can generate a network with a given degree distribution and a given clustering coefficient value between that of LESF networks and that of SFR networks. This is shown in Fig. 4(d). Furthermore, for data of the WLESF model, there is a power law in $C(A)$, and more clear in the cases of $\lambda=3.0$ and $\lambda=5.0$, since in these cases,

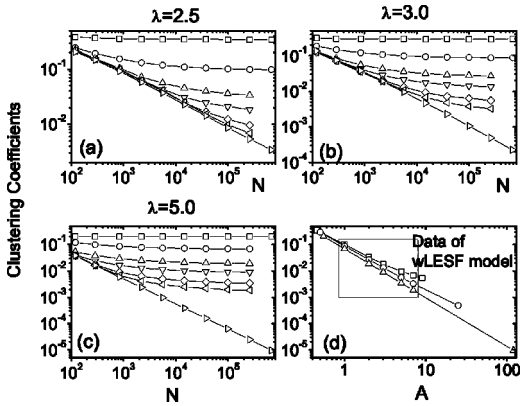

FIG. 4. Clustering coefficients of LESF, WLESF, and SFR model. Lines in graphs (a), (b), and (c) from up to down are the LESF model (squares), WLESF model with $A=1,2,3,5,7$, and SFR model (right toward triangles). (d) Clustering coefficients C vs the cutoff parameter A in the WLESF model, in a log-log scale, for $N=260\,000$ (the last data of WLESF model in (a), (b), and (c)) and $\lambda=2.5,3.0,5.0$ for squares, circles, and triangles, respectively; the first and last data of each line are that of the LESF model and SFR model, respectively. The values of A for these points are adjusted to fit the curve.

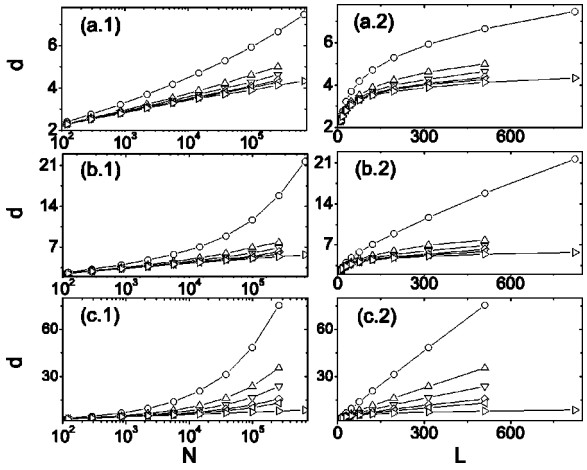

FIG. 5. Average distances of WLESF and SFR model for $\lambda = 2.5$ (a), $\lambda = 3.0$ (b), and $\lambda = 5.0$ (c). In each graph, the lines from up to down are the WLESF model with $A = 1, 2, 3, 5, 7$, and SFR model. The left parts have linear-log scales with x-axis depicts N, while the right parts have linear-linear scale with x-axis depicts lattice side length L or $\sqrt{N}$. Comparisions could be made to see that there are transitions of $d \sim \ln N$ to $d \sim L$ when λ goes larger and A goes smaller.

as Figs. 4(b) and 4(c) show, the network size 260 000 seems large enough for clustering coefficients to be independent of network size.

Figure 5 shows the average distances of the models. Instead of $d \sim \ln N$ (or even $\ln \ln N$ [13]) for various small world models (as SFR model, shown in the graph), for large λ and large N, the average distance of WLESF model obeys a law of $d \sim \sqrt{N}$ (or $d \sim L$, where L is the side length of the lattice). Considering an ideal network that is clustered uniformly, for each cluster, it has N_c nodes, a square area with

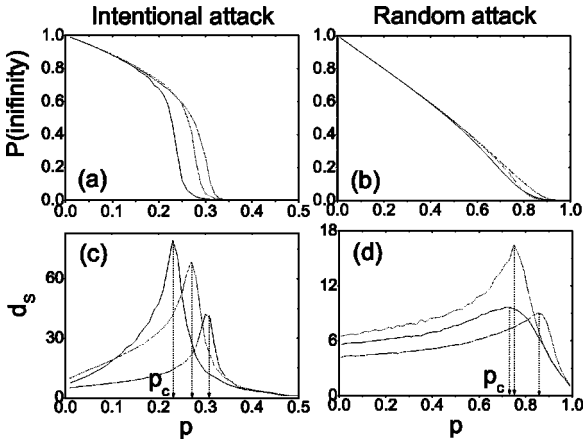

FIG. 6. Properties of percolations on LESF (solid lines), WLESF (dashed lines, $A = 1$ here), and SFR (dashed-dotted-dotted lines) networks when a fraction p of the nodes are removed. For all the three models, $N = 10000$, $\lambda = 3.0$ and $\langle k \rangle = 8$. The left part [(a),(c)] is of intentional attacks and the right part [(b),(d)] is of random attacks. P(infinity) is the proportion of the largest cluster during attacks to the original size of the network; d_S is average distance of the largest cluster (or spanning cluster in terms of percolation theory).

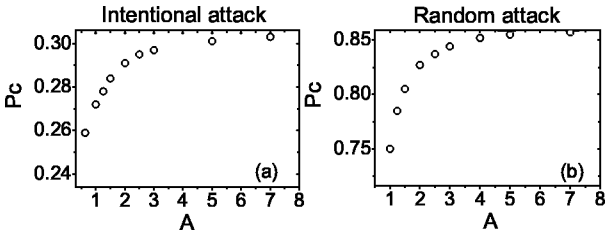

FIG. 7. Percolation transition point p_c vs cutoff parameter A. The upper and lower boundaries indicate the values of p_c of SFR and LESF model, respectively. (a) is for intentional attacks and (b) is for random attacks. For each data, $N = 10000$, $\lambda = 3.0$, and $\langle k \rangle = 8$.

edge l_c and an average network distance d_c, for simplicity it is in addition assumed that two adjacent clusters have only one common node, and each cluster has eight neighbors, as that in a rectangular lattice. Periodic boundary conditions are applied. Suppose that the network has $(2n+1)^2$ clusters, with spatial width $L = (2n+1)l_c$. A simple computation would yield the average distance of the network to be $d = (d_c/3) \times (4n^2 + 10n + 3)/(2n+1)$. Note that for n large, we have $d \sim \frac{2}{3} d_c n$, or $d \sim \frac{1}{3} d_c L$, or else in terms of N, $d \sim \sqrt{N}$. When $n \gg 1$, the boundary conditions can be ignored so the assumption of periodic boundary conditions is not crucial to the result. In case of percolation on SF networks, at the critical point, there appears the same dependence of average network distance on the size of the spanning cluster [14].

Percolations [14–18] on the models are also considered. As shown in Fig. 6, there is a correlation of clustering and the percolation transition point p_c, where p_c is defined as the value of percolation strength when the average distance of the spanning cluster d_S peaks. For both intentional and random percolations, when clustering coefficient is big, as Fig. 3(b) shows, the percolation transition point is small, that is, when a network is heavily clustered, it would be less stable under attacks, either by intentional or by random. Since the WLESF network is tuned by cutoff parameter A, a dependence of p_c on A is expected, and is shown in Fig. 7. Although in case of $\lambda \lesssim 3.0$, there would be no uncommon p_c for random percolations in the limit of $N \to \infty$ [18], for a given network size, considering what factors influence percolation may be useful to understand the network structure.

In Ref. [10], a cutoff distance for each node with degree k is set by $A_c \sqrt{k}$. In the WLESF model, from the connection probability $f(r)$, a cutoff radius R depending on the degree of

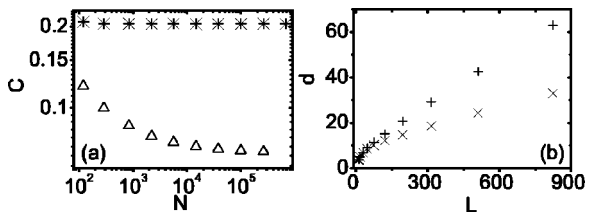

FIG. 8. Effects of cutoff A_c in the LESF model, for $\lambda = 5.0$ (+ for $A_c = 3.5$, $\times$ for $A_c = 7.0$). (a) The clustering coefficient C vs networks size N, triangles are the WLESF model with the same value of λ and $A = 1$, used for comparison. (b) The average network d distance vs the side length of the lattice L.

the central node and A is posed by $\int_1^R dr 2\pi r f(r) = b$, where b is a real number and close to 1. For b equals 0.999 and for large k, R is almost $A\sqrt{7k}$ ($\sqrt{7}$ is not the accurate number, since b could be actually smaller). So $A_c \approx \sqrt{7}A$, and analysis about cutoff lengths and applications in [10] would be also applicable in this WLESF model, with substituting $\sqrt{7}A$ for A_c. Although both of the models have local regions for a node to connect, as set by cutoff radius, the two models are different in the sense that, for a given node, it connects to its nearest neighbors for the LESF model, but for the WLESF model, it connects to its neighbors in the region almost randomly. In this sense, A_c influence little to the clustering property but much to the network distance, as Fig. 8 shows, but A both.

In summary, we have studied a linking weighted SF network embedded on a 2D Euclidean lattice and compared its statistical properties with the LESF and the SFR networks. Using computer simulations, the statistical properties of these models are investigated. It can be seen that the WLESF model gives a smooth transition from LESF to SFR. It means that we can control the degree distribution (LESF, SFR can also), the network distances (LESF can) and the clustering coefficients in the model. Then the model provides more possibilities for approaching real networks with certain required properties.

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